Scheduling Parallel Jobs Under Power Constraints

Kunal Agrawal

Washington University in St. Louis

PARALLEL JOBS: DYNAMICALLY UNFOLDING DAG

- Node: Unit work task.
- Edge: Dependence between tasks.

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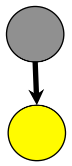

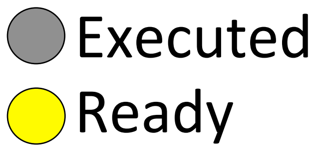

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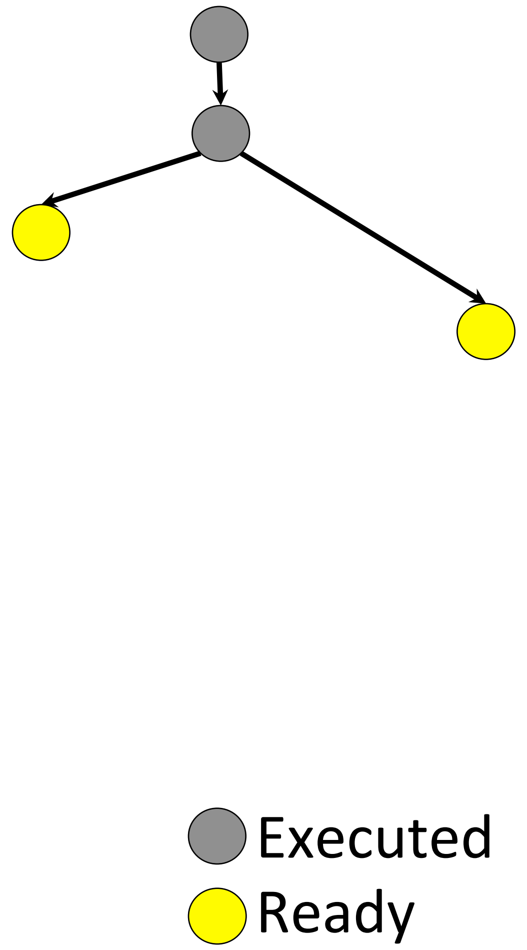

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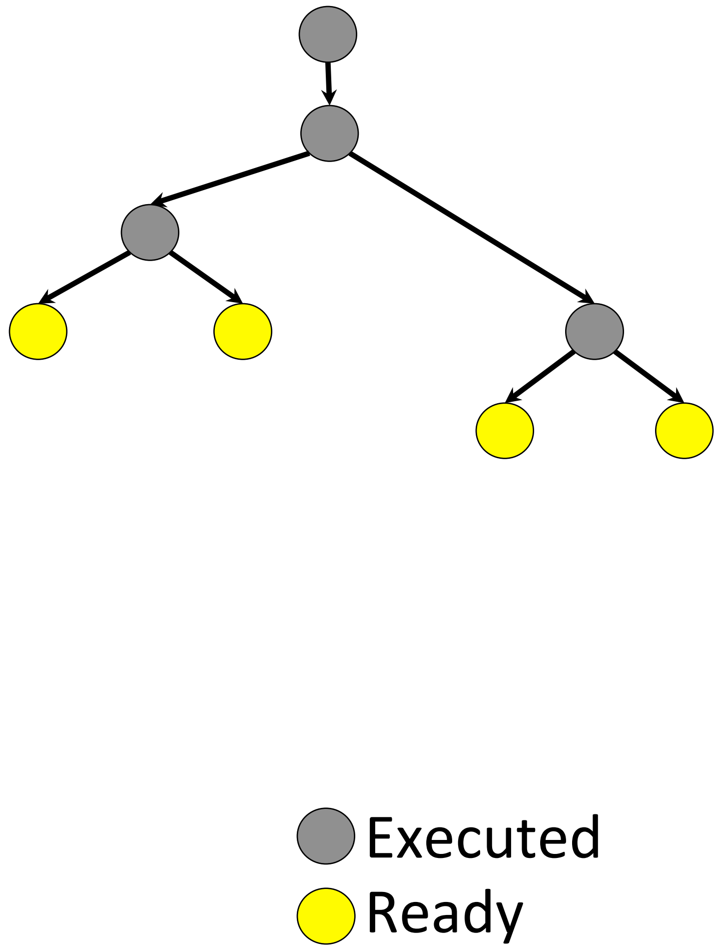

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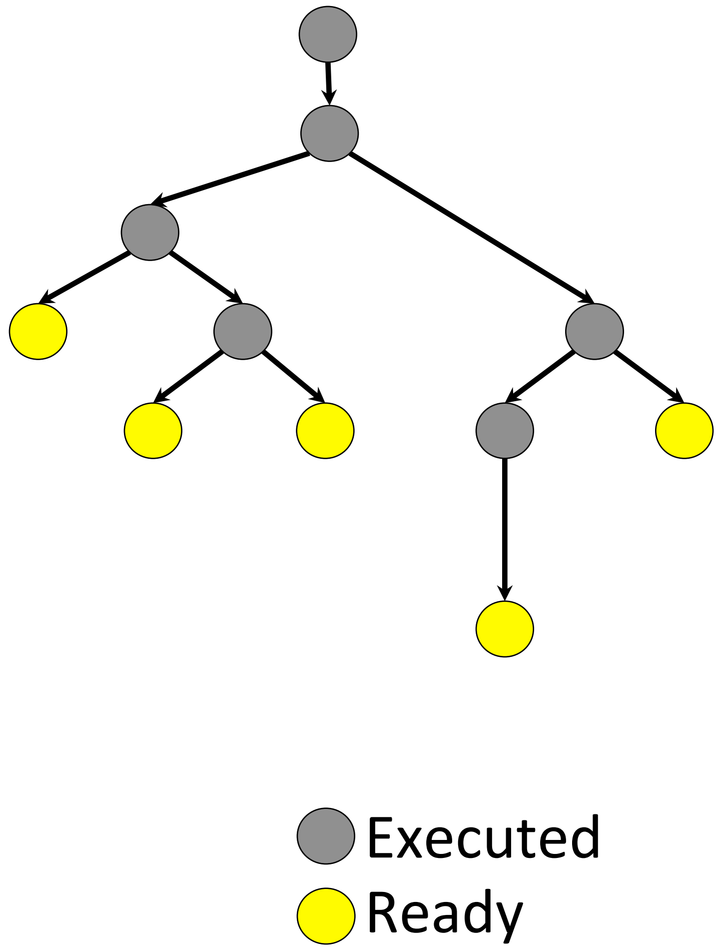

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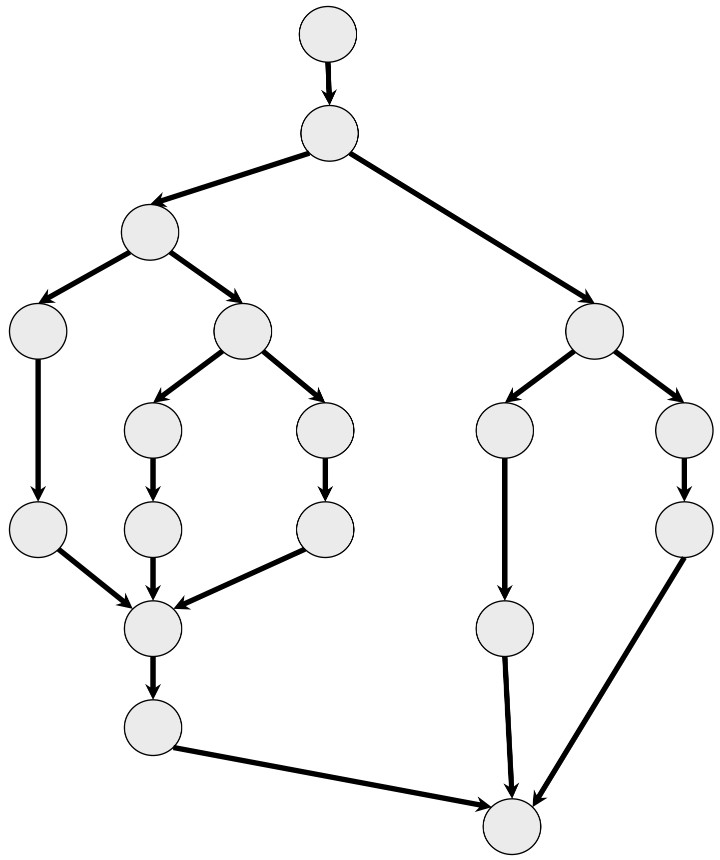

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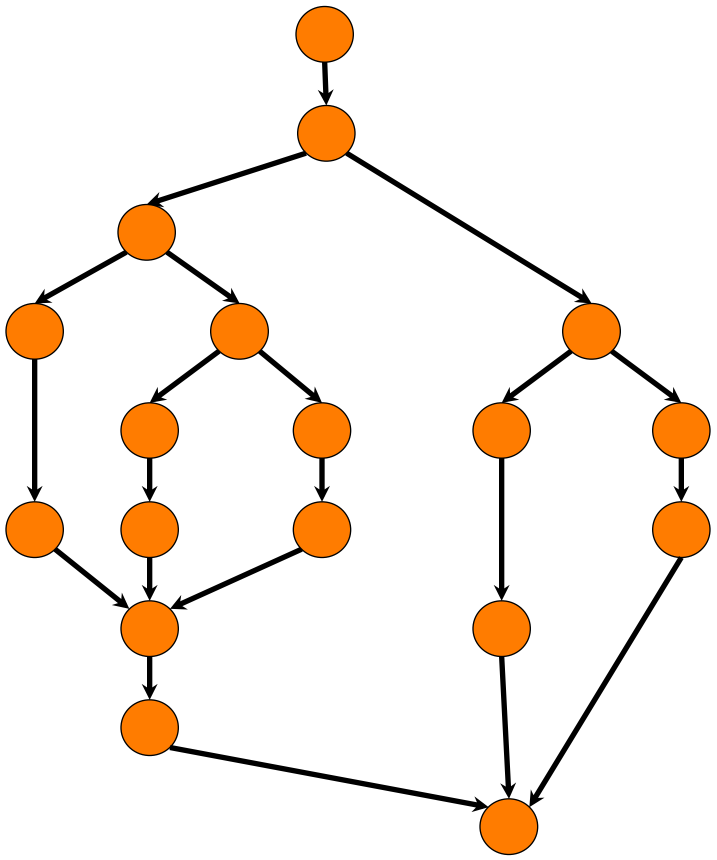
- *W: Work* = Total number of nodes



$W = 18$

PARALLEL JOBS: DYNAMICALLY UNFOLDING DAG

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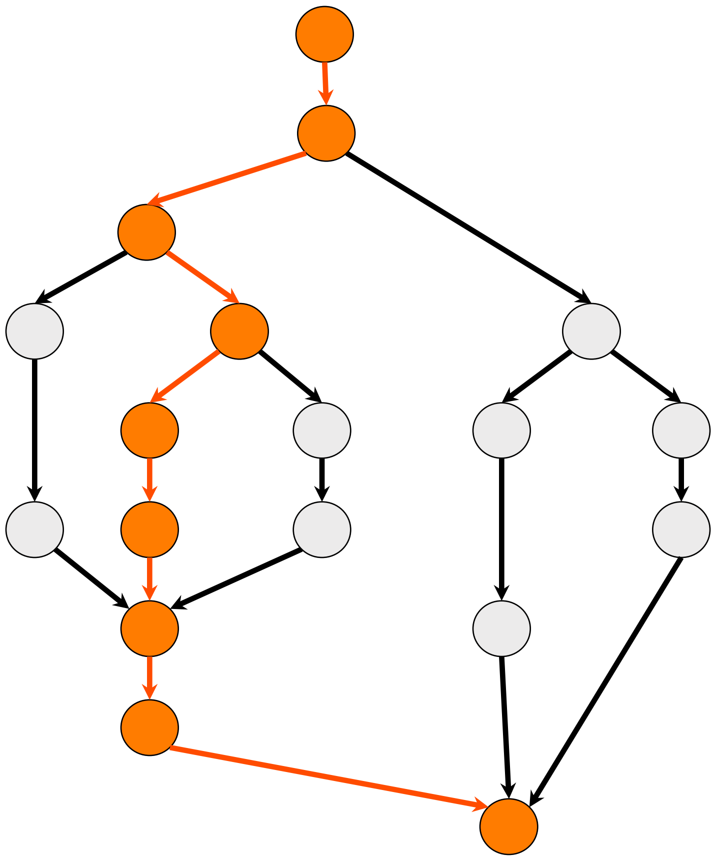

$W = 18$
$S = 9$

Parallel Jobs: Dynamically Unfolding DAG

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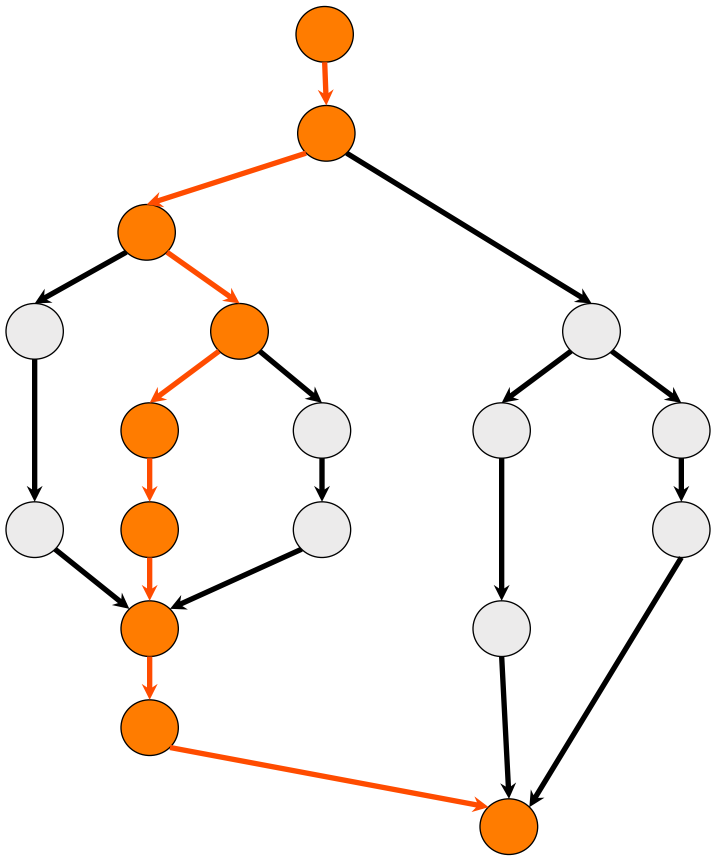
- **Assumption:** We do not know the work, span or the structure of the DAG in advance.



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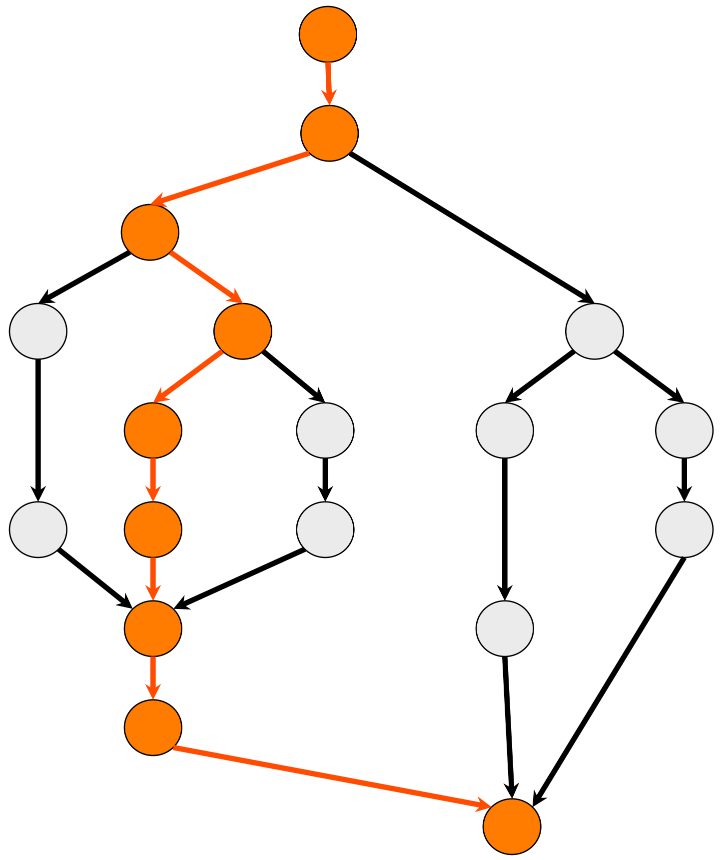
- **ASSUMPTION:** We do not know the work, span or the structure of the DAG in advance.

- With m processors of speed f, list scheduling guarantees a makespan of $W/fm + S/f$ (within 2 of optimal).



$W = 18$
$S = 9$

Power Constraint

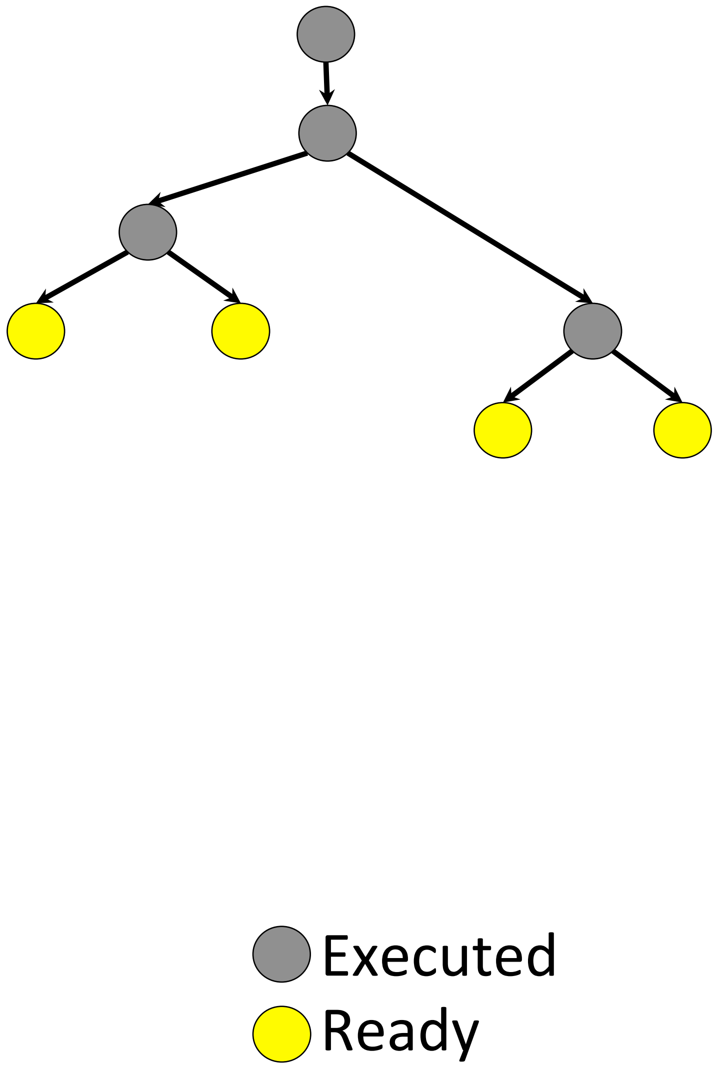
- At any time, we can only use power *P,* but can turn on or turn off processors.
- m processor running at speed f use power $P = mf^{\alpha}$ ($\alpha > 1$).
- With increasing m, the speed of individual processor (f) decreases, but you can do more work in each time step.
- m_{max}=maximum number of processors.
- We are allowed to change the configuration as the job executes, but fewer configuration changes is better.

m	*f*	*mf*
1	10	10
2	7.07	14.14
3	5.77	17.31
4	5	20
5	4.47	22.36

P=100, α=2, assuming all processors run at same speed.

PROBLEM DEFINITION

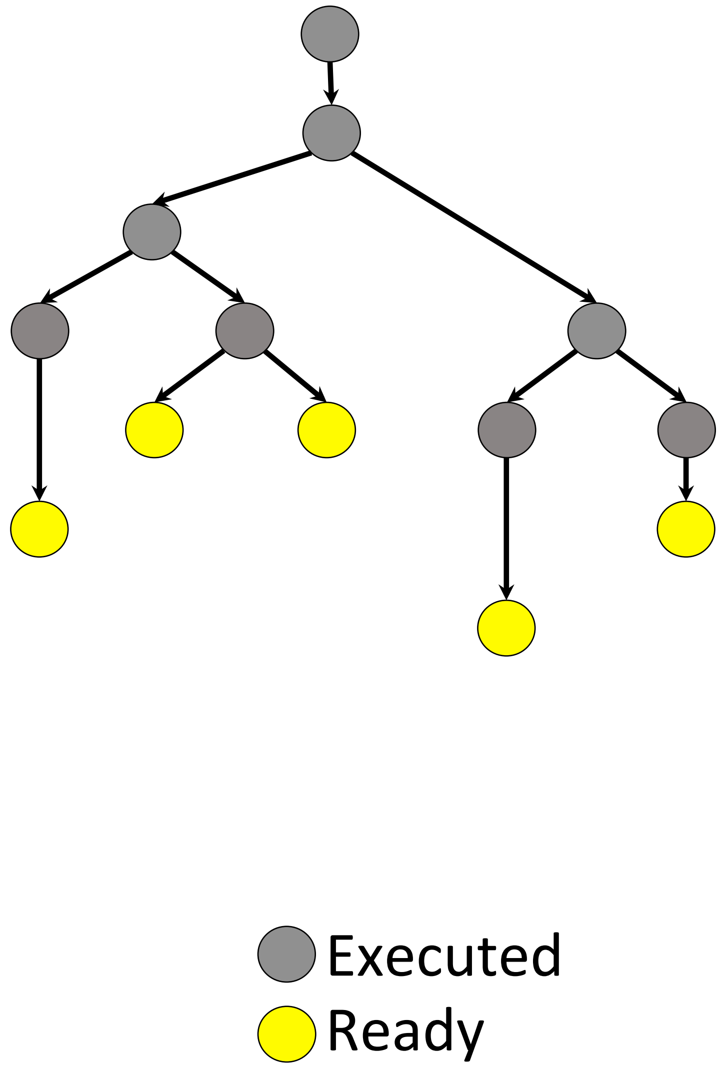
- **INTUITION**: We want to turn the maximum number of processors we can use.



PROBLEM DEFINITION

- **INTUITION**: We want to turn the maximum number of processors we can use.
- With *S* configuration changes, we get
  - optimal makespan if m_{max}> maximum width,
  - about 2-competitive otherwise.

- Question: What is the minimum number of configuration changes to get O(1)-competitive makespan?



The Home-Away-Pattern Set Feasibility Problem

Dirk Briskorn[1]

[1]Bergische Universität Wuppertal, Lehrstuhl für Produktion und Logistik

Single Round Robin Tournament

- $2n$ teams
- $2n-1$ rounds
- each team plays each other team exactly once
- each team plays exactly once per round

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1	2	3	4	5
1-2	1-3	1-4	1-5	1-6
5-4	2-4	3-5	4-6	5-2
3-6	5-6	2-6	2-3	4-3

Scheduling SRRTs

- Scheduling SRRTs by First-break-then-schedule
  - First, the venue of each team in each round is fixed
  - Second, matches are arranged by pairing home-teams and away-teams
- Home-Away-Pattern Set Feasibility Problem: Given the venue for each team in each round, is there a corresponding SRRT?

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	1	2	3	4	5
1	H	H	H	H	H
2	A	H	H	H	A
3	H	A	H	A	A
4	A	A	A	H	H
5	H	H	A	A	H
6	A	A	A	A	A

Home-Away-Pattern Set Feasibility

- Some obvious necessary conditions
  - Number of away-teams has to equal number of home-teams in each round.
  - There must not be two identical home-away patterns.
- Some less obvious ones
  - Miyashiro R., Iwasaki H., Matsui T. (2003) Characterizing Feasible Pattern Sets with a Minimum Number of Breaks. In: Burke E., De Causmaecker P. (eds) Practice and Theory of Automated Timetabling IV. PATAT 2002. Lecture Notes in Computer Science, vol 2740. Springer, Berlin, Heidelberg (for minimum number of breaks)
  - B. (2008): Feasibility of home-away-pattern sets for round robin tournaments, Operations Research Letters, Vol. 36, No. 3, pp 283-284.

Home-Away-Pattern Set Feasibility

	1	2	3	4	5
1	H	H	H	A	H
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The Routing Open Shop Problem: Some Open Problems

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Novosibirsk, Russia
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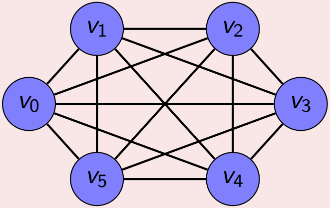
Informal introduction to the Routing Open Shop Problem

The combination of OPEN SHOP and Metric TSP



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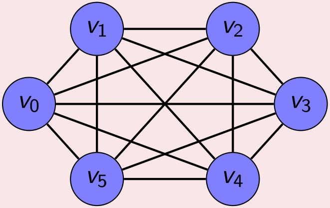

$\{J_1, \ldots, J_n\}$

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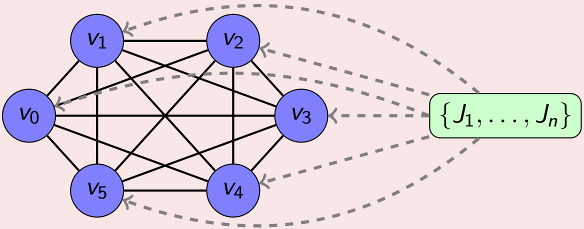

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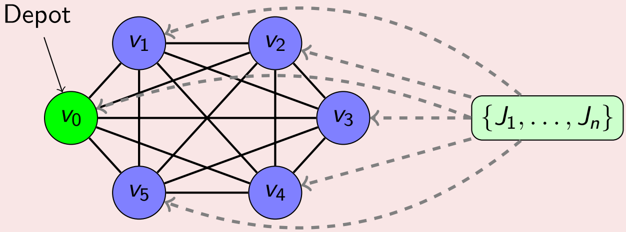

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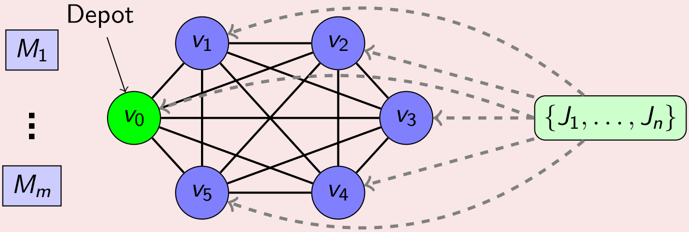

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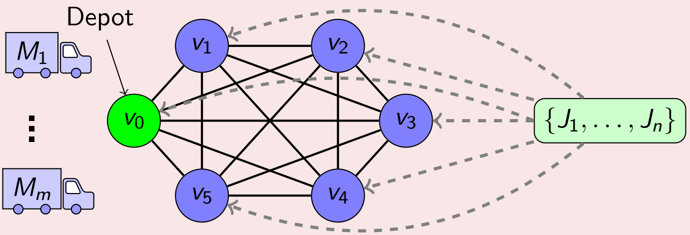

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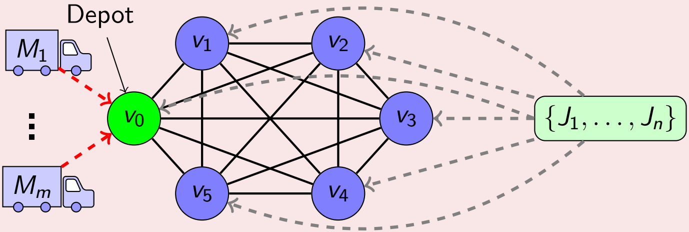

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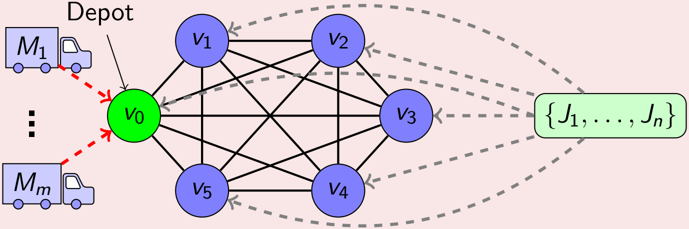

- p_{ji} — processing time of the operation of job J_j and machine M_i;
- $G = \langle V, E \rangle$ — transportation network;
- τ_{kl} — travel time between v_k and v_l;
- $R_i(S) = \max\limits_{k} \left(\max\limits_{J_j \in \mathcal{J}_k} C_{ji}(S) + \tau_{0k} \right)$;
- $R_{\max}(S) = \max R_i(S) \to \min_S$ — the makespan.

Lower bound

- $\ell_i = \sum_{j=1}^{n} p_{ji}$ — load of machine M_i,
- $d_j = \sum_{i=1}^{m} p_{ji}$ — length of job J_j,
- $\ell_{\max} = \max \ell_i$ — maximal machine load,
- $d^k_{\max} = \max_{J_j \in \mathcal{J}_k} d_j$ — maximal length of job from v_k,
- $\Delta^k = \sum_{J_j \in \mathcal{J}_k} d_j$ — total load of vertice v_k,
- T^* — length of the shortest route over G (TSP optimum)

Standard lower bound

$$\bar{R} = \max\left\{\ell_{\max} + T^*, \max_k \left(d^k_{\max} + 2\tau_{0k}\right)\right\}$$

Open Problems (not a complete list)

Routing open shop

Some Known Facts

1. NP-hard even for $\langle m = 2, G = K_2 \rangle$ (and a bunch of polynomially solvable classes for that case).
2. For general case best known approximation algorithm is $O(\log m)$-approximate.

Open Problems

1. Is there an *const*-approximation for general case?
2. Consider function $F(m) = \sup_{I \in \mathcal{I}_m} \frac{R^*_{\max}(I)}{\bar{R}(I)}$. Is $F(m)$ bounded by any constant?

Routing open shop with preemptions

Some Known Facts

1. Problem with $\langle m = 2, G = K_2 \rangle$ is polynomially solvable (and $R^*_{\max} = \bar{R}$).
2. Problem with $G = K_2$ is strongly NP-hard if m is a part of input.
3. Problem with $\langle m = 2, G = K_3 \rangle$ is polynomially solvable IF for some node $\Delta^k > \bar{R} - 2\tau_{0k}$.

Open Problems

1. Complexity of
$\langle m = 2, G = K_3 \rangle$,
$\langle m = 3, G = K_2 \rangle$,
$\langle m = 2, G = K_{const} \rangle$,
$\langle m = const, G = K_2 \rangle$ cases.

The 13th Workshop on Models and Algorithms for Planning and Scheduling Problems (MAPSP 2017)

Delayed-Clairvoyant Scheduling

Sorrachai Yingchareonthawornchai, Eric Torng
Michigan State University, MI, USA

15 June 2017

MICHIGAN STATE
UNIVERSITY

Delayed-Clairvoyant Scheduling

$$1|online-time-clv, pmtn, r_j|\sum F_j$$

Shortest Remaining Processing Time (SRPT) is optimal

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Any deterministic algorithm is $\Omega(n^{\frac{1}{3}})$ competitive

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Shortest Elapsed Time First (SETF) is $(1+\epsilon)$ speed $1+\dfrac{1}{\epsilon}$ competitive

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$1|online - time - delayed - clv, pmtn, r_j|\sum F_j$ Delay factor $\alpha \in [0, 1]$

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Shortest Elapsed Time First (SETF) is $(1+\epsilon)$ speed $1 + \frac{1}{\epsilon}$ competitive

$1|online - time - delayed - clv, pmtn, r_j|\sum F_j$ Delay factor $\alpha \in [0, 1]$

For $\alpha < 1$ SETF+SRPT is $\frac{1+\alpha}{1-\alpha}$ competitive

Open problems

- Formalize non-uniform delay factor reduction
- Get an analogous result for clairvoyance when $\alpha < 1$
  - Weighted Flow time:

HDF is $(1+\epsilon)$ speed $1 + \frac{1}{\epsilon}$ competitive

Is WSETF+HDF $(1+\epsilon)$ speed $1 + \frac{1}{\epsilon}$ competitive for $\alpha = \frac{1}{1+\epsilon}$?

A Chair’s Scheduling Problem

Samir Khuller

University of Maryland

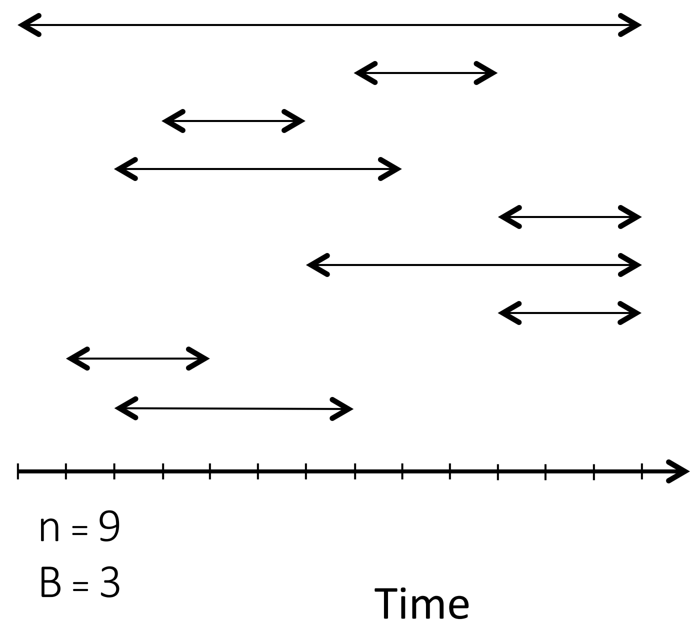
Research supported by NSF CCF 0937865, CCF 1217890 and Google.

Whats the problem?

- Alumni, Companies, Deans office, Provost's office, unhappy students, PhD students, faculty candidates, hiring meetings, faculty, staff....
- Lots of hour long meetings (some multiple)!
- Meetings from 9am to 6pm only
- GOAL: Maximize number of days at home!

Scheduling to Minimize Active-Time

- *n* jobs
  - release times and deadlines
  - length
- batch machine
  - time is slotted
  - in each slot, “active” or “inactive”
  - “active” slot → can schedule ≤ B jobs
- minimize number of “active” slots



Scheduling to Minimize Active-Time

- *n* jobs
  - release times $r_i \in Z$ and deadlines $d_i \in Z$
  - length ℓ_i
- batch machine
  - time is slotted
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  - “active” at t → can schedule ≤ B jobs
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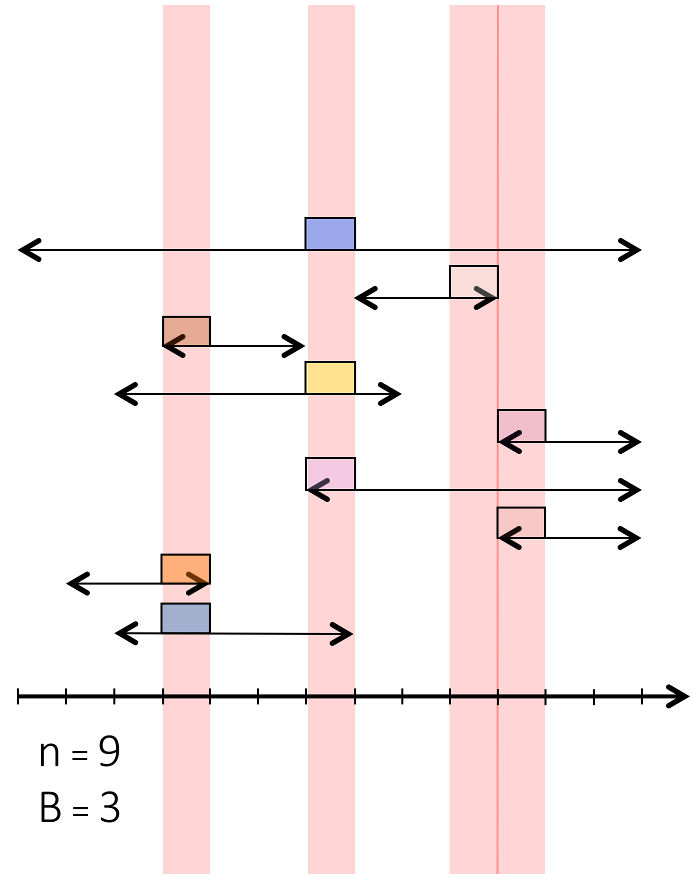

FOUR ACTIVE SLOTS

Lets focus on UNIT length jobs for now

Batching Algorithms

- Wolsey's greedy algorithm [Wolsey, 1982]
  - O(log n)-approximation
- Exact alg via Dynamic Programming
  [Even, Levi, Rawitz, Schieber, Shahar, Sviridenko, 2008]
  - Time complexity: $O(n^2 T^2 (n+T))$
- Faster exact algorithm?
- Also models taxi drop offs to the train station from Dagstuhl.

Lazy Activation

- *n* jobs
  - release times and deadlines
  - length
- batch machine
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  - “active” slot → can schedule ≤ B jobs
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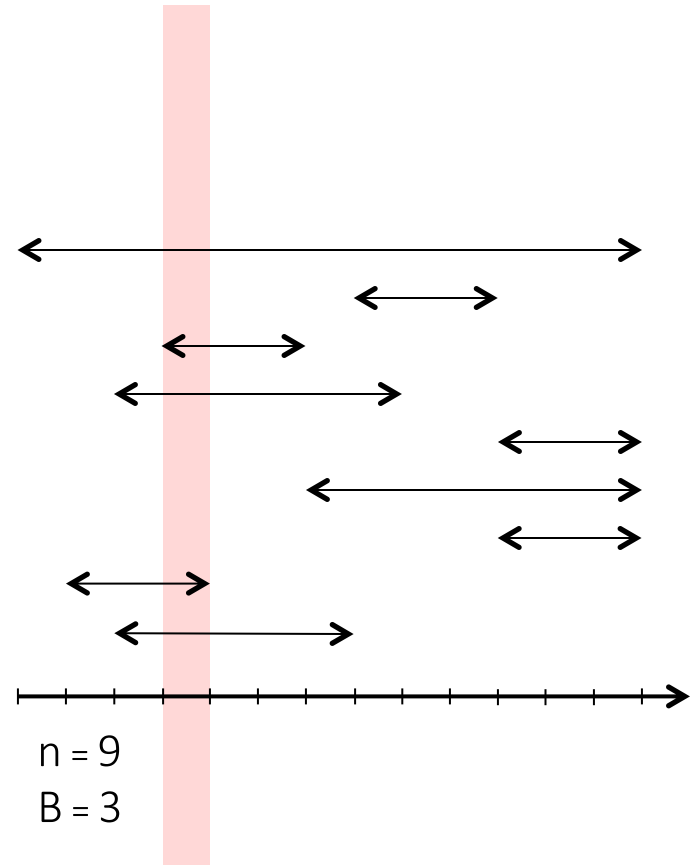
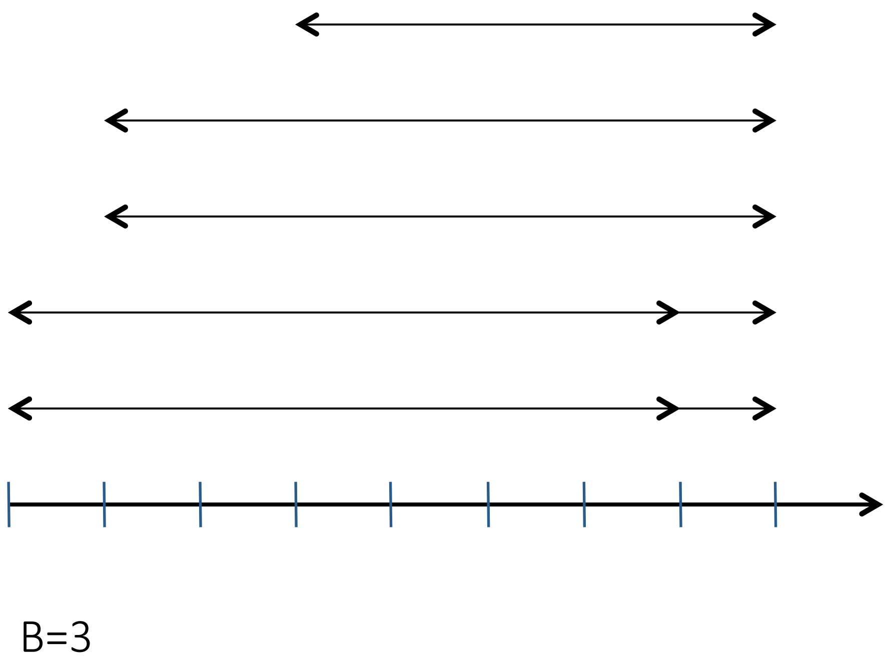


Each column is a set of capacity B

Lets focus on UNIT length jobs for now

Lazy Activation [Chang, Gabow, K., 2012]

- **Step I.** Scan slots right to left, and decrement deadlines in overloaded slots
  - favor decrementing deadlines of jobs with earlier release times

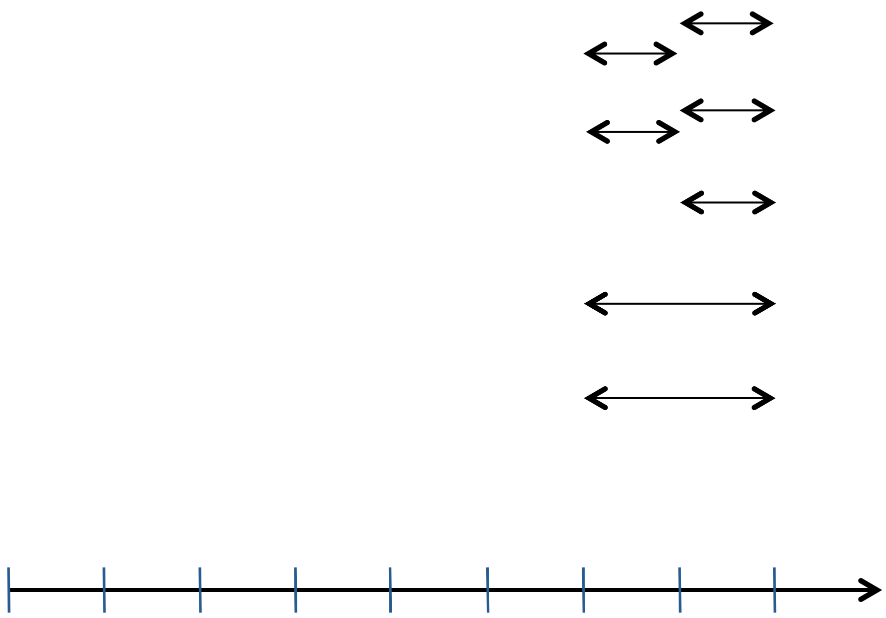


Lazy Activation [Chang, Gabow, K., 2012]

- **Step I.** Scan slots right to left, and decrement deadlines in overloaded slots
  - favor decrementing deadlines of jobs with earlier release times
- **Step II.**
  - Order jobs s.t.
  - Consider deadlines LTR:
    - Schedule at any outstanding jobs with deadline
    - Fill the remaining capacity with feasible jobs of later deadline, favoring those with earlier deadline

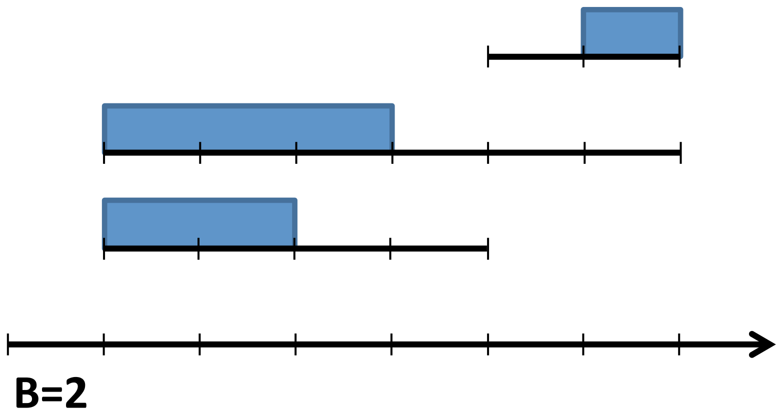
Lazy Activation Maximizes Throughput

- On infeasible instances, Step I preserves the maximum number of jobs

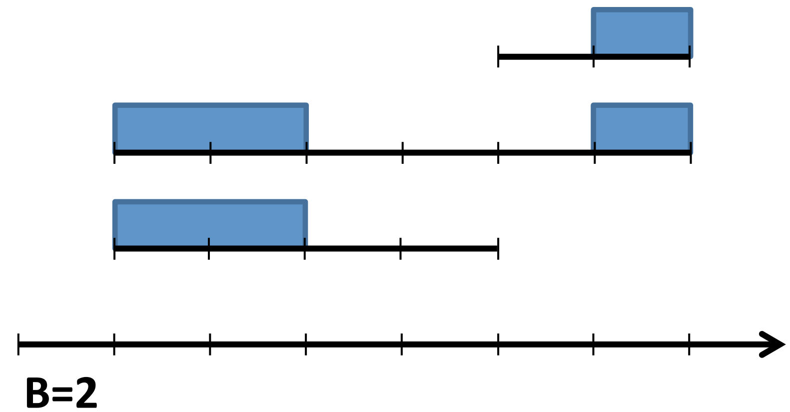


B = 3

Arbitrary-length Jobs [Chang,K,Mukherjee SPAA 2014]



active time: 4



active time: 3

NON-PREEMPTIVE:
NP-hard via 3-PARTITION

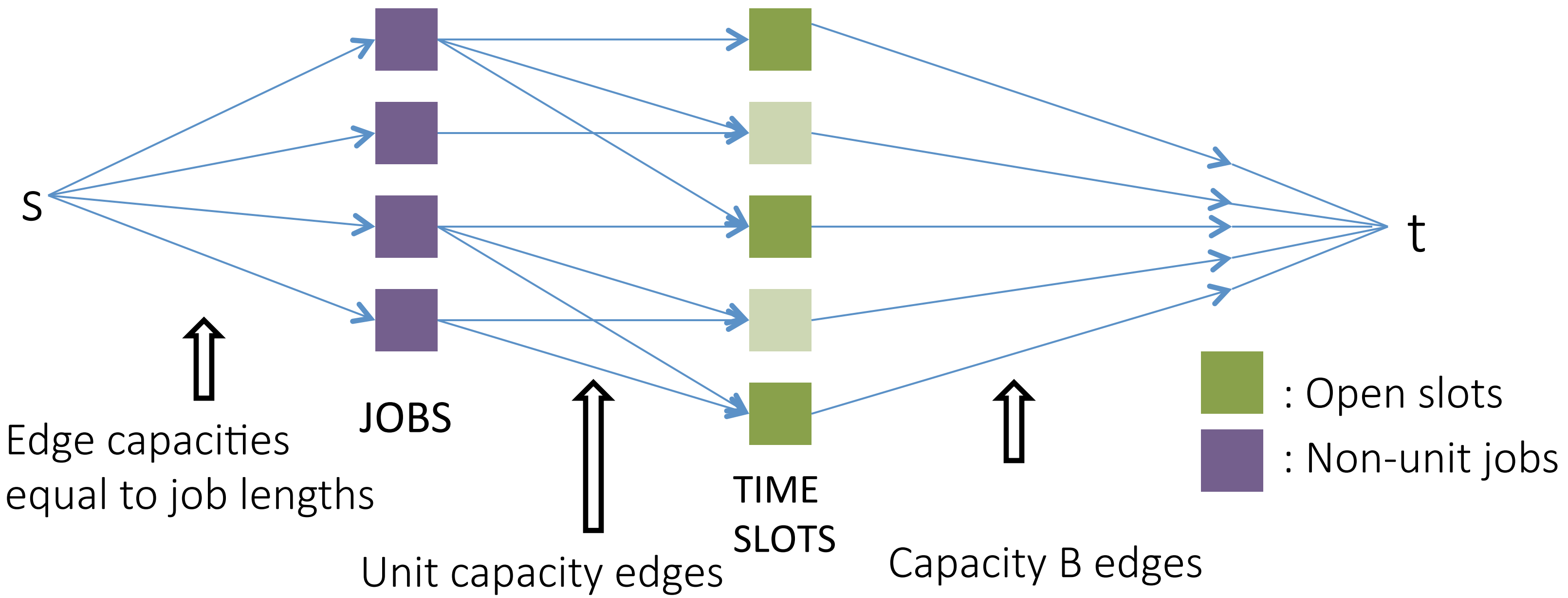
PREEMPTIVE complexity?

We have a 3 approximation (BusyTime)

We have a 2 approximation

Relation with max-flow

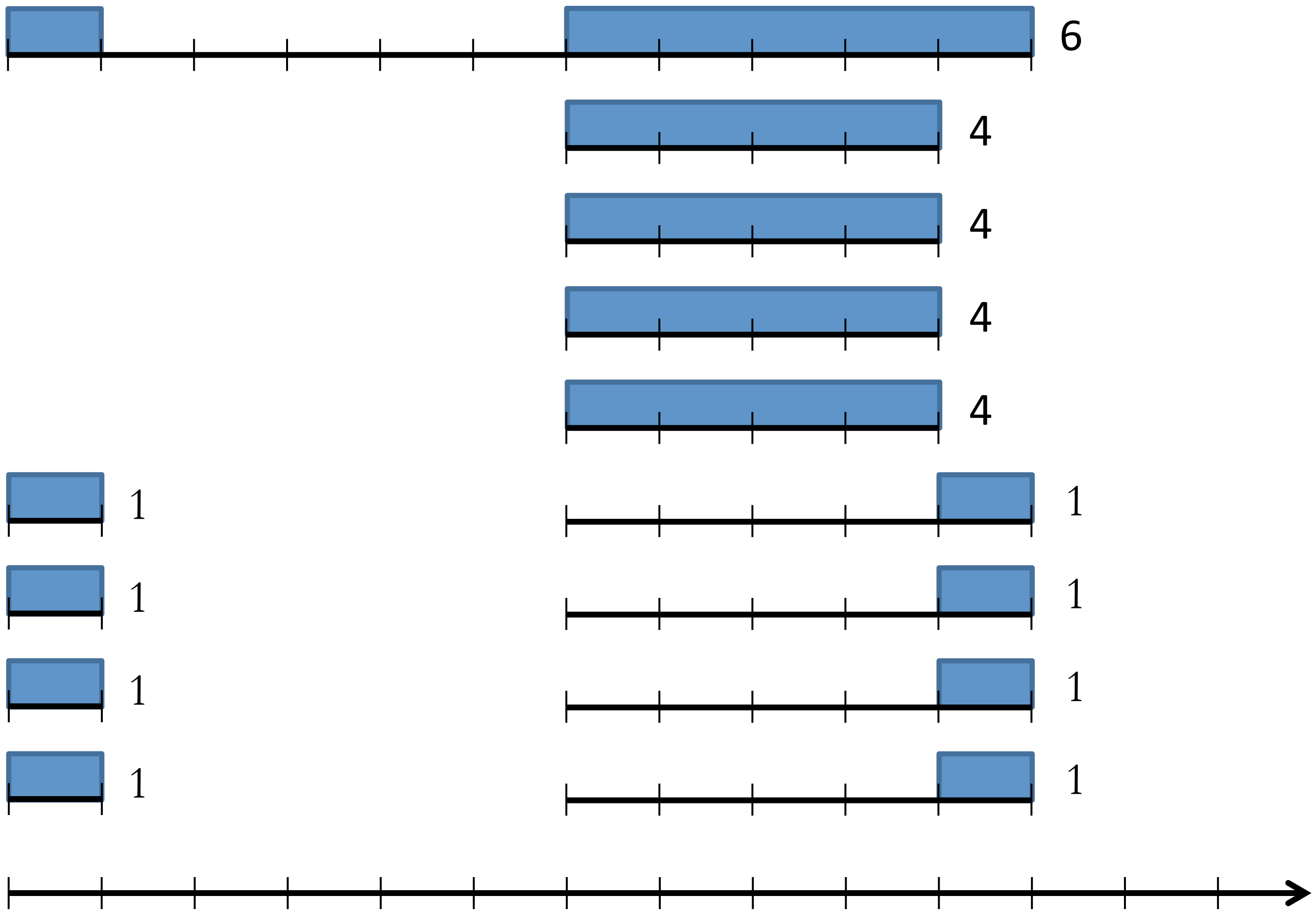
- Cost of a solution: number of open or active slots.
- Observation: Given a set of integrally open slots, max-flow will find a feasible integral assignment of jobs, if there exists one.
- This follows from max-flow integrality theorem.



Minimal Feasible Solutions

- Getting job assignments from a set of active slots: network flow computation
- Minimal feasible solutions (MFS): shutting down any active slot → infeasible
- Start from all slots being active, as long as a feasible schedule is possible, close a slot

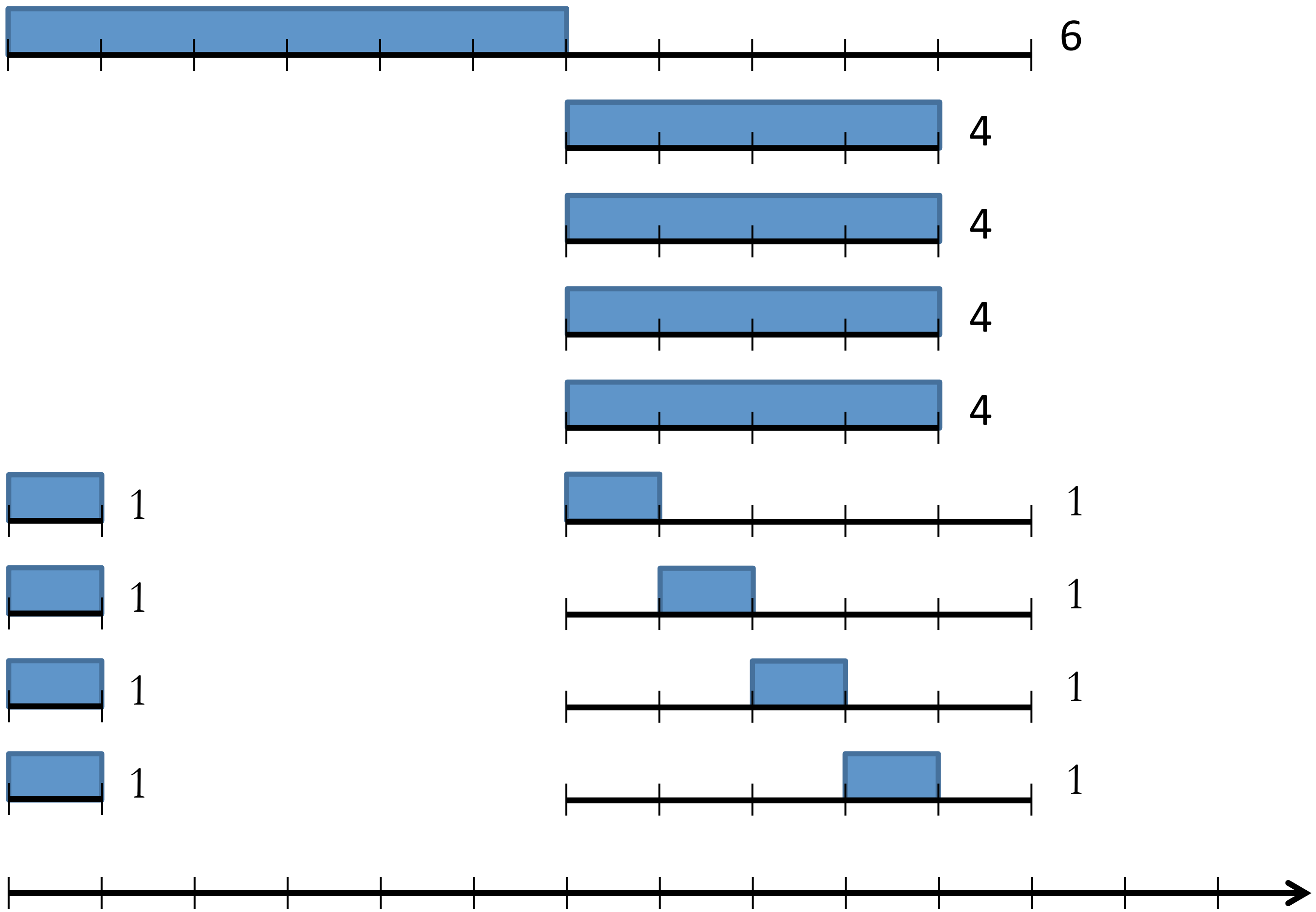
Minimal Feasible Solutions



B=5

OPTIMAL SOLUTION. COST=B+1

Minimal Feasible Solutions

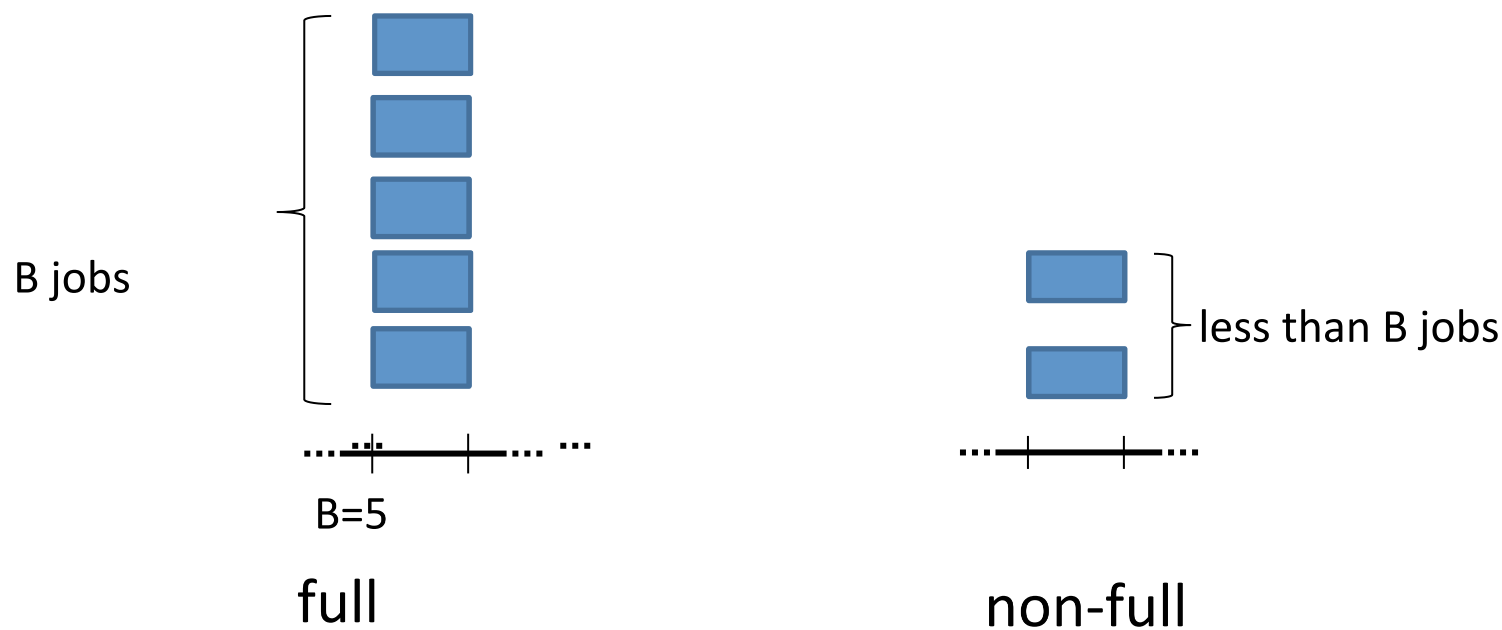


B=5

A MFS SOLUTION. COST=2B

Every MFS is 3-approximate

- Every MFS can be “left-shifted”
- Dichotomy of active slots



LP rounding based algorithm

$$\min \sum_{t \in T} y_t$$

$$\text{s.t. } x_{t,j} \le y_t \quad \forall\, j \in J,\ t \in T$$

$$\sum_{j \in J} x_{t,j} \le B\, y_t \quad \forall\, t \in T$$

$$\sum_{t \in [r_j, d_j]} x_{t,j} \ge p_j \quad \forall\, j \in J$$

$$0 \le y_t \le 1, \forall\, t \in T$$

$$x_{t,j} \ge 0, \forall\, t \in [r_j, \ldots, d_j]$$

What does the LP give?

- Factor 2 approximation (tight).
- [Kumar-Khuller] MFS that shuts slots left to right also gives factor 2 approximation.
- Local-Search is not optimal but might be <2.
- Still do not know if its NP-complete..

Parallel Machine Scheduling with Weighted Completion Time Objective and Online Machine Assignment

Sven Jäger

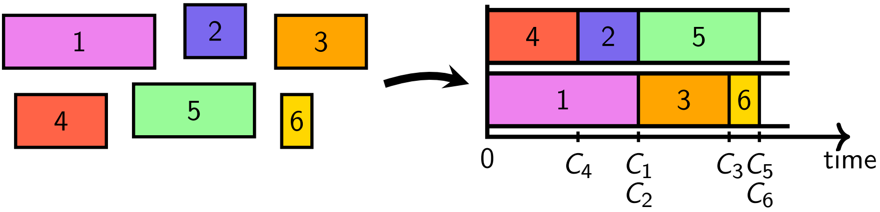
Combinatorial Optimization
and Graph Algorithms
Technische Universität Berlin

MAPSP Open Problem Session
13 June 2017

$P||\sum w_j C_j$

Given: Jobs with processing times $p_j \geq 0$ and weights $w_j \geq 0$, $j = 1, \ldots, n$ and number m of machines

Task: Process each job non-preemptively for p_j time units on one of the m machines such that the total weighted completion time $\sum_{j=1}^{n} w_j C_j$ is minimized.



WSPT Rule

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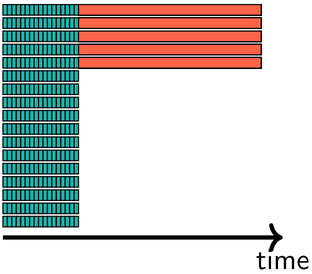
1. Sort jobs by non-increasing ratios w_j/p_j.
2. Do list scheduling in the obtained order.

Theorem [KK86] The WSPT rule has performance guarantee $\frac{1+\sqrt{2}}{2} \approx 1.207$.

Worst case instance: $w_j = p_j$ for all j.



Online Machine Assignment

- Jobs arrive sequentially and must immediately be assigned to the machines.
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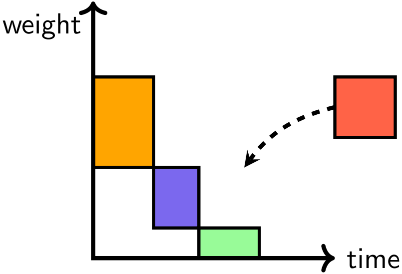

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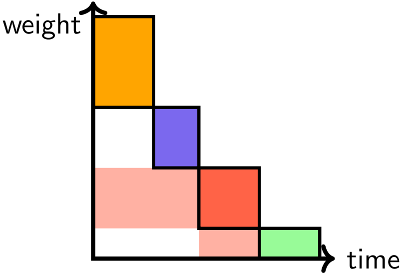

Competitive ratio of MinIncrease

Observations

- If jobs arrive ordered by decreasing w_j/p_j, $\textsc{MinIncrease}$ does the same as list scheduling.

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Open Question

Is $\textsc{MinIncrease}$ always $\frac{1+\sqrt{2}}{2}$-competitive?

Appendix

Competitive Ratio for Stochastic Counterpart

Theorem (MUV06)

The algorithm that assigns each job to the machine with minimal increase of expected weighted completion time is $1 + \frac{(m-1)(\Delta+1)}{2m}$-competitive, where Δ is an upper bound on the coefficient of variation of the processing times.

References

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$P||C_{\max}$ in time $f(\overline{p}) \cdot (n + \log m)^{O(1)}$ for some function f

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